*Chapter Three*

# Statements and Conditionals

## 3.1 STATEMENTS AND COMPOUNDS

### 3.1.1 STATEMENTS

What is a statement? As mentioned in chapter 1, this apparently innocent question can lead us quickly into deep philosophical waters. A statement cannot be just a sentence, since sentences have a great variety of linguistic functions. We use them not only to make statements, but to ask questions, issue commands, make exclamations, exhortations, and so forth. But these other functions are by and large identifiable by their grammatical moods, interrogative ("Are you going?"), and Imperative ("Go!," "Let's go!"). So why not say a statement is a sentence (in indicative or perhaps subjunctive mood) used to make an assertion, i.e., a *declarative sentence*? This is close. But the same statement is made by the following three different declarative sentences: "It is raining," "Il pleut," and "Es regnet," the second and third being French and German for the first. And if I say "It is raining" twice in a row, have I made two statements or one? Is "Sean Penn was divorced by Madonna" a different statement than "Madonna divorced Sean Penn"?

These difficulties can be resolved as follows. First we need to distinguish a *type* of sentence from a *token* of the same sentence. Thus a declarative sentence such as "It is raining" is one that could be uttered or written on many different occasions by different people. Each such utterance or act of writing is a token of that sentence-type. Second, among types of sentence we may distinguish the declarative *sentence* (which, when written down or spoken, is subject to grammatical rules), from the *proposition* (what is expressed by a declarative sentence, its content or meaning). Thus "It is raining" is one sentence, and tokens of it may be uttered on any number of occasions. "Il pleut" and "Es regnet" are different sentences, but express the same proposition. Likewise "Sean Penn was divorced by Madonna" and "Madonna divorced Sean Penn": statements in the passive mode ("S was divorced by M") express the same proposition as corresponding active mode statements ("M divorced S"), and from here on will always be so reinterpreted.

Now clearly statements in the sense of utterances are no concern of logic; nor are sentences insofar as they are mere grammatical items. Consequently logic has been said to deal with statements in the last of these senses only, types of statements that are propositions: hence the traditional name for the logic of statements, "Propositional Logic." In the twentieth century, however, the notion of propositions came in for trenchant criticism on the grounds that no one has been able to give an adequate account of when two propositions could be regarded as identical. As a result, most logic texts define a statement as "a sentence that is either true or false." But there are difficulties with this definition too, as noted above and discussed further in the GLOSSARY under STATEMENT; see also PROPOSITION.

Fortunately, we need not resolve this philosophical issue in order to proceed. The definitions already given in chapter 1 of statement and proposition are adequate for our purposes:

> A *statement* is a sentence (or part of a sentence) that expresses something true or false.

But to say a statement is true or false is just to say what it asserts is true or false. Thus in this sense, i.e., semantically, a statement may be identified with what it expresses, namely the proposition, which we defined as follows:

> A *proposition* is what a statement expresses as true or false.[1]

In contrast to this, a question may ask whether something is true or false, but it does not express something that *is* true or false. Similarly, a command such as "Do up your shoe laces!" does not assert anything true or false; nor does an exhortation such as "Let's go, Blue Jays!" Finally a supposition such as "Suppose you are right" does not itself assert the statement "you are right," although the latter statement is contained within the supposition.

### 3.1.2 COMPOUNDS

As we saw above, one of the things that these definitions take into account is that propositions are often expressed by parts of sentences. We saw that in a statement like Augustine's "If no one asks me, I know [what time is]," two other statements are contained in it:

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[1] This definition goes all the way back to Chrysippus of ancient Athens: "A proposition is what can be asserted or denied on its own, for example, 'It is day' or 'Dion is walking.' The proposition gets its name [*axioma*] from being accepted [or rejected]; for he who says 'It is day' seems to accept [*axioun*] that it is day" (Brad Inwood and L.P. Gerson, *Hellenistic Philosophy* [Indianapolis: Hackett, 1988], p. 84).

"no one asks me" and "I know what time is." They are called *components* of the original statement. But not all parts of statements that can be said to express something true or false can be regarded as components. Take for example the following statement, from Umberto Eco's medieval thriller *The Name of the Rose*:

(1) Benno admitted that his enthusiasm had carried him away.[2]

"Benno admitted that" and "his enthusiasm had carried him away" are both parts of a sentence, and each could be taken to express a proposition, so they are statements by the above definition. But the first does not appear to be a component of the whole statement in the right sense. For the second, we could substitute any other statement at all, and still get something meaningful: "Benno admitted that nylon tights tear easily," for example. Try doing that with the first! This motivates the following definitions:

One statement is a *component* of another if substituting it within the original by any other statement whatever still yields a meaningful statement.[3]

A *compound statement* is any statement that contains one or more component statements.

A *simple statement* is any statement that has no components.

To see how this works, consider the following statement:

(2) Bacon was right in saying that the conquest of learning is achieved through the knowledge of languages. (*ibid*., p. 191)

"Bacon was right in saying that the conquest of learning is achieved" is a statement, as are "learning is achieved through the knowledge of languages," "Bacon was right in saying that," and even "Bacon was right." But none of them are components of the original statement.[4] Substituting the third of these, for example, by "Nylon tights tear easily" yields "Nylon tights tear easily the conquest of learning is achieved through the knowledge of

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[2] Umberto Eco, *The Name of the Rose*, p. 158. All the statements in this section are taken from this book.

[3] This definition of component statement is due to Copi and Cohen, *Introduction to Logic* (11th edition), p. 301.

[4] Substituting the first of these by "Nylon tights tear easily" yields "Nylon tights tear easily through the knowledge of languages"! You may be able to imagine a context in which this is meaningful; but the original statement must be meaningful when any statement is substituted. Try substituting, say, "Nothing tastes better than lychees."

languages," which is pure gibberish. "The conquest of learning is achieved through the knowledge of languages," however, is a component statement. (Try substituting other statements for it.)

### 3.1.3 STATEMENT OPERATORS

In Statement Logic we are not concerned with compound statements of all kinds, but only a very delimited class of them. The point can be made by looking at some further examples from Eco's *The Name of the Rose*:

(3) "I began to think [that] I had encountered a forgery."
(4) "Ubertino could have become one of the heretics he helped burn, or [he could have become] a cardinal of the holy Roman church."

Both of these are compound statements according to our definition. In the first, any statement at all could have been substituted for "I had encountered a forgery" and the result would still be a meaningful statement. Similarly in the second, any statements could have been substituted for "Ubertino could have become one of the heretics he helped burn" and "he could have become a cardinal of the holy Roman church," with a similar result. A good way of thinking of this is to imagine the words or phrases "I began to think that" and "or" as *operators* that work on statements to produce other statements. Such operators are usually called *statement connectives* (not a very good term, since the ones that operate on single statements do not "connect" them to anything!). We will call them *statement operators*.

A *statement operator* is a word or phrase which operates on a statement or statements to form a compound statement.

Statement operators that form a compound by operating on only *one* statement are called *unary*; those that join together a *pair* of statements are called *binary*.

Thus the phrases "I believe that," "It must be concluded that," "It is not the case that" are all examples of *unary operators*: they turn *single* statements into other single compound statements. On the other hand, "...or...," "...and...," and "from...it follows that...," are *binary operators*: each joins a *pair* of statements into a compound statement.

Now of all the myriad possible statement operators there are a few that hold special interest for us in statement logic. Naturally these include the inference indicators we have already encountered, that is, the premise- and conclusion-indicators: "Consequently...," "Therefore...," "It follows that...," "...since...," and so forth. But in addition there are those that typically join together statements into the compound statements that constitute the premises and conclusion of commonly occurring single-inference arguments.

There are precisely five of these operators that are basic to logical reasoning. They are:

- "It is not the case that __"
- "if __ then __"
- "__ and __"
- "__ or __," and
- "__ if and only if __."

These five are distinguished from other phrases used to make compound statements by the fact that the *truth value*—that is, the truth or falsity—of any compound formed by them is a function only of the truth or falsity of their component statements. That is, for each combination of the truth values of the components there will be a unique truth value—true or false—of the compound. Consequently, these five operators are called *truth-functional operators*.

> A *truth-functional operator* is one that forms a compound statement whose truth value is a function of the truth values of the component statements.
>
> All other statement operators are (surprise!) *non-truth-functional*.

This distinction is best illustrated through examples. Statement (1) above is a compound statement formed from the phrase "Benno admitted that__" operating on the component statement "his enthusiasm had carried him away." So "Benno admitted that__" is a unary operator. But when we prefix it to some other statement *p*, the truth value of the resulting compound statement "Benno admitted that *p*" depends on whether *p* is one of those statements that Benno admits! Contrast this with the compound formed from prefixing "[Benno's] enthusiasm had carried him away" by the unary operator "It is not the case that__":

(5) "It is not the case that Benno's enthusiasm had carried him away."

Here if the component is true, the compound is clearly false; and if the component is false, the compound is true; and *this is so for any statements that we care to prefix with this operator*. This is what makes "It is not the case that__" a truth-functional operator. Similar (but not so trivial) considerations apply to the four binary truth-functional operators. We shall investigate how their compounds' truth values vary with the components' in a later chapter.

We have already seen how it is convenient to abbreviate statements by capital letters. In the same way it helps to have abbreviations for the five truth-functional operators. So we introduce special symbols for them, writing

| | | | |
|---|---|---|---|
| $\neg A$ | for | not A (it is not the case that A) | |
| $A \rightarrow B$ | for | if A then B | |
| $A \,\&\, B$ | for | A and B | |
| $A \vee B$ | for | A or B | and |
| $A \leftrightarrow B$ | for | A if and only if B | |

Here A and B stand for certain individual statements—*any* statements, compound or simple. Thus in $A \rightarrow B$, statement A could be a compound statement. For instance, take the following statement by the ancient Greek Antiphon:

> (6) If someone were to bury a bed and the rotting wood came to life, it would become not a bed, but a tree.—*A Presocratics Reader*, p. 105

Here if A is "someone buries a bed and the rotting wood comes to life," and we symbolize "it becomes not a bed, but a TREE" by T, this gives $A \rightarrow T$.[5] But A is itself a compound of two statements "someone is BURYING a bed" (B) and "the rotting wood came to LIFE" (L). So we may symbolize it as $B \,\&\, L$. This would make the whole statement

$$B \,\&\, L \rightarrow T$$

Looking at this formula, you may be wondering why you suddenly feel hungry. But you should also notice that it is ambiguous. If you plug back in the component statements, you could get the different statement

> (7) Someone is burying a bed, and if the rotting wood were to come to life, it would become not a bed, but a tree.

This statement asserts that someone is actually burying a bed, whereas Antiphon was making no such claim. The ambiguity of formulas like $B \,\&\, L \rightarrow T$ is easily removed by using parentheses, just as we do in arithmetic and algebra. The formula $3 + 2/5$ can be disambiguated by distinguishing $(3 + 2)/5$ from $3 + (2/5)$: these yield the different values 1 and 3.4. Similarly Antiphon's statement (6) should be symbolized

(F6) $(B \,\&\, L) \rightarrow T$

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[5] Here I am tacitly converting the subjunctive and conditional tenses into indicative mood (and also back again). I should warn, however, that there is a large literature on subjunctive and counterfactual conditionals which I am thereby ignoring.

whereas statement (7) is symbolized

(F7) $B \& (L \rightarrow T)$

Here I am using the notation (F6), for example, for the formula symbolizing statement (6). I have also introduced the *convention that the capital letter symbolizing each individual statement will be the first letter of the capitalized word* (sometimes part-word) *in that statement*. I will follow this convention from here on in this book. (I will make sure, however, that different statements are symbolized by different letters.) Thus the formula symbolizing statement (5), "It is not the case that Benno's ENTHUSIASM had carried him away," is

(F5) $\neg E$

Note that we do not symbolize statement (5) by E, just because it is a negative statement, a denial. This invokes a further convention regarding symbolizing. This is that *all statements A, B, C,... are to be positive assertions*, rather than denials. Again analogously with algebra, the unary operator '¬' does not need parentheses. There is nothing to be gained in clarity by writing 'Not E' as $\neg(E)$, rather than just plain $\neg E$. Nor would there be any gain in symbolizing "It is not true that something NEW had not occurred" by $\neg(\neg N)$, as opposed to plain $\neg\neg N$. On the other hand, parentheses are necessary for the negation of E & F, that is, $\neg(E \& F)$, to distinguish it from $\neg E \& F$.

Finally, there is no gain in clarity from writing parentheses around the outside of the whole of a compound statement standing by itself, e.g., by writing (F6) as

(F6*) $((B \& L) \rightarrow T)$

Nevertheless, we can understand such parentheses as being there implicitly in (F6) without having to write them in. This means that it is neater to think of the binary operators as *always* introducing parentheses (or equivalent groupers, like [brackets] and {braces}), with the understanding that the outermost ones do not need to be written in explicitly. All of this can be summed up neatly by the following set of rules:

Rules of statement formation

(i) (*unary operator*) if A is a statement, then ¬A is also a statement.

(ii) (*binary operators*) if A and B are both statements, then (A * B) is also a statement, where '*' stands for any of the binary operators '&', '∨', '→', and '↔'. Likewise, so are [A * B] and {A * B}.

(iii) (*convention regarding outermost groupers*) in any compound formed by the binary operators, the outermost groupers are understood, rather than explicitly written in: e.g., {A * [B * C]} is written A * [B * C].

One final point: you should appreciate that some paraphrase may be necessary when symbolizing. Thus "Emily is here, and so is Fred" would be understood as "EMILY is here, and FRED is here."

Now let's look at some examples, taken from John Horgan's irreverent and rambunctious interviews with scientists and intellectuals in his book *The End of Science*:

(a) If in fact it [physics] were just a social CONSTRUCT, it would have FALLEN apart long before this.—Steven Weinberg (p. 74) $C \rightarrow F$
This is a simple "if…then…" statement. We'll be looking at these in the next section. The binary operator '→' introduces parentheses, but since these are outermost, we regard them as understood.

(b) If I didn't spend my life CONCENTRATING on string theory, I would simply be MISSING my life's calling.—Edward Witten (p. 68) $\neg C \rightarrow M$
Here the "if-clause" is a negative statement.

(c) If there's going to be such a TOTAL theory of physics, in some sense it couldn't conceivably [have] the CHARACTER of any theory I've seen.—Roger Penrose (p. 179)
Here the "then-clause" is a negative statement. $T \rightarrow \neg C$

(d) My position in linguistics is a MINORITY position, and it ALWAYS has been.—Noam Chomsky (p. 150) $M \mathbin{\&} A$
This a simple conjunction of M and A, where A is understood to be the statement "My position in linguistics has ALWAYS been a minority position."

(e) If you've got a BEAR by the tail, there comes a point at which you've got to LET it go, and STAND back.—Thomas Kuhn (p. 46) $B \rightarrow (L \mathbin{\&} S)$
Here the "then-clause" is a compound of "there comes a point at which you've got to let it GO" and "[there comes a point at which you've got to] STAND back."

(f) I have opinions that I DEFEND rather vigorously, and then I find out how SILLY they are, and I GIVE them up!—Paul Feyerabend (p. 50) $D \mathbin{\&} (S \mathbin{\&} G)$

(g) Evolution no doubt OCCURS, and it's been SEEN to occur, and it's occurring NOW.—Lynn Margulis (p. 129) $O \mathbin{\&} (S \mathbin{\&} N)$

(f) and (g) could have as well been symbolized by $(D \mathbin{\&} S) \mathbin{\&} G$ and $(O \mathbin{\&} S) \mathbin{\&} N$, but *not* by $D \mathbin{\&} S \mathbin{\&} G$ and $O \mathbin{\&} S \mathbin{\&} N$, or by $(D \mathbin{\&} S \mathbin{\&} G)$ or $[O \mathbin{\&} S \mathbin{\&} N]$, etc.

(h) Before it was very easy. Either you believe in Jesus CHRIST, or you believe in NEWTON.—Ilya Prigogine $C \vee N$
Here C := "you believe in Jesus CHRIST," N := "you believe in NEWTON."

SUMMARY

- One statement is a **component** of another if substituting it within the original by any other statement whatever still yields a meaningful statement.
- A **compound statement** is any statement that contains one or more component statements.

- A **simple statement** is any statement that has no components.
- A **statement operator** is a word or phrase which operates on a statement or statements to form a compound statement.
- Statement operators that form a compound by operating on only one statement are called **unary**; those that join together a pair of statements are called **binary**.
- A **truth-functional operator** is one that forms a compound statement whose truth value (truth or falsity) is a function of the truth values of the component statements.
- The five truth-functional operators we shall be concerned with are:

| not-A | if A then B | A and B | A or B | and | A if and only if B. |
|---|---|---|---|---|---|
| $\neg A$ | $A \rightarrow B$ | A & B | $A \vee B$ | | $A \leftrightarrow B$ |

**Symbolizing conventions:**

- The capital letter symbolizing each individual statement will be the first letter of the capitalized word (sometimes part-word) in that statement.
- We use the letters A, B, C, . . . to symbolize positive assertions, rather than denials.
- A compound formed by one of the binary operators comes with groupers, e.g., parentheses—$(A \rightarrow B)$, (A & B), $(A \vee B)$, $(A \leftrightarrow B)$—but the outermost groupers of a compound statement are understood, rather than written explicitly, e.g., $C \vee (A \rightarrow B)$.

## EXERCISES 3.1

1. *State whether each of the following statements* from Michael Ondaatje's *The English Patient* is *simple or compound*. *Identify the component statements in each compound statement*.
   (a) She heard a far grumble of thunder. (p. 62)
   (b) If a man leaned back a few inches he would disappear into darkness. (p. 143)
   (c) In Canada pianos needed water. (p. 63)
   (d) We stood up at the end and you walked off the table into his arms. (p. 53)
   (e) He is a writer who used pen and ink. (p. 94)
   *Example:*
   (e) "He is a writer" is a statement, but is it a component statement? If we substitute for it, say, "Malebranche was born in Paris," we get "Malebranche was born in Paris who used pen and ink," which is not meaningful. So it is not a component statement of (e), which is therefore simple.

2. *Identify any statement operators in the following statements* by various scientists quoted by John Horgan in *The End of Science*. In each case *identify whether it is truth-functional* or *non-truth-functional*, and whether it is *binary* or *unary*.

(a) [Bohm] insisted that reality was unknowable. (p. 90)
(b) We still live in the childhood of mankind.—John Archibald Wheeler (p. 83)
(c) As we have discovered more and more fundamental physical principles, they seem to have less and less to do with us.—Steven Weinberg (p. 73)
(d) If Edward Witten is a philosophically naive scientist, Weinberg is an extremely sophisticated one . . . (p. 72)
(e) Since I know a little bit about global economic models, I know they don't work! —Philip Anderson (p. 210)
(f) The situation cannot declare itself until you've answered the question.—John Archibald Wheeler (p. 82)
(g) It's a very deep position, but I also think it's very deeply wrong.—Stephen Jay Gould (p. 136)

*Example:*

(g) 'But' is a *binary* operator, connecting "It's a very deep position" with "I also think it's very deeply wrong." Since the whole statement can't be true if either of the components is false, 'but' is *truth-functional*. "I also think it's very deeply wrong" is also a compound statement, though, formed from the *unary* operator 'I also think [that]' and "it's very deeply wrong." 'I also think [that]' is clearly *non-truth-functional*.

3. Symbolize the following statements from Eco's *The Name of the Rose*:
(a) "If from this conjunction a BABY was born, the infernal RITE was resumed." (p. 63)
(b) "Providence did not WANT futile things glorified." (p. 127)
(c) "If the window had been OPEN, you would immediately have THOUGHT he had thrown himself out of it." (p. 29)
(d) "The DAYS of the Antichrist are finally at hand, and I am AFRAID, William!" (p. 66)
(e) "In the BEGINNING was the Word, and the Word was WITH God, and the Word was GOD." (p. 3)
(f) "I could sit at the TABLE with the monks, or, if I were EMPLOYED in some task for my master, I could stop in the KITCHEN." (p. 105)
(g) "If it was STIRRED properly and promptly, it would remain LIQUID for the next few days, thanks to the cold climate, and then they would make BLOOD puddings from it." (p. 77)
(h) "Adelmo THREW himself of his own will from the parapet of the wall, struck the ROCKS, and SANK into the straw." (p. 103)
(i) "If Adelmo FELL from the east tower, he must have GOT into the library, someone must have first STRUCK him so he would offer no resistance, and then this person must have found a way of CLIMBING up to the window with a lifeless body on his back." (p. 103)

## 3.2 CONDITIONAL STATEMENTS

Anyone who has watched or listened to a baseball game will be familiar with this kind of post-game analysis:

(1) If he makes the CATCH, they're OUT of the inning.

This is about a crucial play that happened hours before. Translated out of baseball-speak into English it means

(2) If he had made the CATCH, they would have been OUT of the inning.

(But of course, he didn't; the opposing team capitalized on his error and scored seven unearned runs.) Baseball-speak has no sense of the subjunctive whatever. It also lacks tenses, expressing all actions, past, present, or future as taking place in a kind of timeless present. Now this may be a regrettable impoverishment of language, but it does have the merit of making clear what follows from what. And since that's what we're concerned with in logic, we do exactly the same here when we symbolize statements. We make no distinction between the two statements above, symbolizing them both as

(F1) $C \rightarrow O$

In fact the same formula would symbolize a great variety of English sentences:

(3) *Had* he made the catch, they would have been out of the inning.
(4) His having made the catch *implies* they'd be out of the inning.
(5) *Should* he make the catch, they'll be out of the inning.
(6) They would have been out of the inning, *if* he had made the catch.
(7) *Provided* he makes the catch, they're out of the inning.
(8) They're out of the inning, *provided* he makes the catch.
(9) His making the catch *will result in* their being out of the inning
(and even, in baseball-speak,
(10) He makes the catch, and they're out of the inning.
—though this way of expressing a conditional is particularly confusing if you've just turned on the radio. Are they saying he made the catch or not?).

(2) through (10) can be re-expressed as statement (1), which is said to be *in standard form*.

Conditional statements are so often involved in logical reasoning that it is convenient to have some terminology for talking about them. The "if-clause"—here "he makes the catch"—is called the *antecedent*, Latin for that which comes before (the arrow). The "then-clause" (in this case the 'then' is tacit) is called the *consequent*—represented here by "they're out of the inning."

The important thing to note here is that the antecedent, despite its name, does not always come first in a natural language statement. It is the statement following the word 'if' (or 'provided that,' or whatever phrase is equivalent to it). It comes first logically, in that it states the *condition* for the other statement's holding. So don't just blindly symbolize "They're OUT of the inning if he makes the CATCH" as O → C: that's wrong. Put the statement in standard form, then symbolize.

The crucial thing we are trying to capture about conditionals in formal logic is the notion of "following from." The one thing we can't have if the consequent follows from the antecedent is for the antecedent to be true and the consequent false. This is what makes the operator '→' truth-functional: it operates on the antecedent and the consequent in such a way that there is no statement of the form "antecedent consequent" with a true antecedent and a false consequent. Such a conditional is called a *truth-functional* or a *material* conditional. The subject of conditionals is complex and philosophically interesting, and whole books have been written on it. In ordinary discourse there is generally a meaning-relationship or some relation of relevance between the antecedent and the consequent. Thus "If you fall down those steps, you will hurt yourself." Such a relationship is not necessary, however, for the truth-functional operator: it simply has to preserve the fact that a true antecedent does not lead to a false consequent.

Of course, not every statement containing the word 'if' is a conditional. (I wondered if you'd noticed. It looks as if you're following all this very well. As if you wouldn't, with your intelligence!) The main culprits seem to be 'ifs' that could be replaced by 'whethers,' and 'ifs' that could be replaced by 'thoughs.' "As if…" and "What if…?" introduce fictional scenarios, and really shouldn't cause you any confusion. The combinations 'even if' and 'only if' require more discussion, though, and I'll come back to these later.

A slightly more insidious bogus conditional is a kind of literary device, which is easier to present by way of example than to explain:

> If Einstein had succeeded in transforming time into space, Gödel would perform a trick yet more magical: He would make time disappear. (Palle Yourgrau, *A World Without Time* [Cambridge: Basic Books, 2005], p. 6)

Obviously it is not being asserted that Gödel's performing this trick is somehow conditional on Einstein's accomplishment, which is in fact being taken for granted here. The 'if' here seems to be roughly equivalent to 'whereas.'

Finally, there are complex conditionals, those whose antecedents or consequents are themselves conditional statements. In symbolizing these, it is prudent to proceed on a step-by-step basis:

(i) Symbolize the component statements.
(ii) Put all the conditionals in their standard "If C then O" form.
(iii) Symbolize the conditionals from the innermost ones outwards.

Here's an example:

If *US Shipyard*'s own bid to acquire the property FAILS, the company would be interested in a LEASE so long as the payments are REASONABLE.

(i) If F, then L so long as (i.e., provided that) R.
(ii) If F, then if R, then L.
(iiia) If F, then (R → L)
(iiib) F → (R → L)

SUMMARY

- A **conditional statement** is a statement asserting that one statement (say, B) is conditional on another (say, A): if A then B. The 'if' statement A is called the **antecedent**, the dependent statement B is called the **consequent**.
- A **truth-functional** or **material conditional** is one such that a true antecedent does not lead to a false consequent.
- **Complex conditionals** are those whose antecedents or consequents are themselves conditional statements.

## EXERCISES 3.2

4. *Which of the following are conditional statements*? *Rephrase any such statements that are not in standard form*.
   (a) We will refund in full if the article is defective.
   (b) Were I to say that, I would be wrong.
   (c) You are looking at me as if you know something.
   (d) If I were a carpenter and you were a lady, would you marry me anyway?
   (e) You may enter provided you are a member.
   (f) Since you ask politely, I will explain.

5. *Symbolize the following conditional statements using the first letter of each capitalized word for the components, which must all be positive statements:*
   (a) If you TRAVEL every path you will not FIND the limits of the soul.—Heraclitus
   (b) If I had a HAMMER, there'd be no more FOLK singers.—comedian Billy Connolly
   (c) If Indonesia does not END the violence, it must INVITE the international community to assist in restoring security.—US President Bill Clinton

(d) Nobody is going to want to continue to INVEST there if they're going to ALLOW this sort of travesty to go on.—US President Bill Clinton
(e) If God did not EXIST, it would have been necessary to INVENT him.—Voltaire
(f) If something EXISTS without any effect at all, its existence is NEGLIGIBLE.—early Buddhist doctrine
(g) Were I the MOOR, I would not be IAGO.—Shakespeare's *Othello*
(h) If there were no CHRYSIPPUS, there would be no STOA.—Stoic philosopher Chrysippus
(i) I am extraordinarily PATIENT provided I get my OWN way in the end.—British PM Margaret Thatcher
(j) If an argument is VALID, then, as long as its premises are TRUE, it is also SOUND.—definitions of validity and soundness
(k) Should the Red Sox SWEEP the Yankees in the weekend series, they will get the WILD card, provided the Blue Jays LOSE tomorrow.
(l) If you think you UNDERSTAND it [quantum theory], that only shows you don't KNOW the first thing about it.—Niels Bohr

6. *Render each formula* (a)-(d) *into a readable English Statement* using the dictionary provided*:*
C := The standings given in the paper are CORRECT.
S := The Red Sox SWEEP the Jays in their weekend series.
W := The Athletics will qualify for the WILD card.
T := The Athletics win TWO more games.
   (a) $S \rightarrow \neg W$
   (b) $C \rightarrow (T \rightarrow W)$
   (c) $S \rightarrow (\neg T \rightarrow \neg W)$
   (d) $C \rightarrow \{S \rightarrow (T \rightarrow W)\}$

7. *Symbolize the following conditional statements* from Shakespeare's *Othello*, Act II, Scene 3, using the first letter of each capitalized word for the components:
(a) IAGO: If I can FASTEN but one cup upon him,
With that which he hath drunk tonight already,
He'll be as full of QUARREL and offense
As my young mistress' dog.
(b) OTHELLO: If I once STIR,
Or do but LIFT this arm, the best of you
Shall sink in my REBUKE.

## 3.3 MODUS PONENS

### 3.3.1 ARGUMENT FORM AND SUBSTITUTION INSTANCE

The ancient Buddhists denied that there is some unchanging substance or matter underlying all the changing qualities we observe. Their view was that since qualities change from one moment to the next, the correct way to express this is to say that there are different things at each different moment. That is, they were committed to the premise that

If qualities are REAL, they are THINGS.[6]

which we may symbolize $R \rightarrow T$. Interestingly, their main opponents, the Sãnkhyas, also agreed with this statement, but had an entirely different view of the world. They denied that qualities were things, and consequently asserted the unreality of the changing qualities that we see, claiming that only eternal matter is real.

In the terminology explained above, then, the Buddhists held the antecedent of the conditional to be true—in logical parlance, they *affirmed the antecedent* of the conditional—and therefore inferred its consequent. That is, they argued:

| | |
|---|---|
| If qualities are REAL, they are THINGS. | $R \rightarrow T$ |
| But qualities are real. | $R$ |
| So they are things. | $\therefore T$ |

It's convenient to have a way of writing the symbolization of a single-inference argument like this all on one line, which we do as follows, separating the premises by commas:

$$R \rightarrow T, R \therefore T$$

This inference is as basic as you can get in logic. All languages have some counterpart to the conditional, and it simply means that the consequent follows from the antecedent. Thus anyone who affirmed a conditional and its antecedent but refused to allow that the consequent followed, could not be said to have understood what a conditional means. In other words, it is impossible to deny the validity of inferring the consequent from a conditional and its antecedent. We can see this by reverting to our definition of formal validity: *no argument of this form can have all true premises and a false conclusion*, that is, the conditional and its antecedent both true and the conclusion false.

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[6] For this and following portrayals of Buddhist logic I am indebted to Shcherbatskoi, F.I., Dharmakirti, and Dharmottara, *Buddhist Logic* (New York: Dover Publications, 1962). The conditional appears on p. 97.

The argument form was well summarized by Chrysippus, a Stoic logician teaching in Athens in the third century BCE, who proposed five basic argument schemata, of which this was the first:

If the first, then the second
The first
———————————
Therefore the second

Here "the first" and "the second" are placeholders for any individual statements. We'll use lower-case letters $p$, $q$, etc., instead. These are called *statement variables*, by analogy with the variables in algebra. They stand for *any* statements, whereas the capital letters we have used to represent individual statements are analogous to constants. Any individual argument having a given form is said to be an *instance* or *substitution instance* of that form. The premises and conclusions of the argument must be *substitution instances* of the corresponding variables. For example, the Buddhists' argument is an instance of the valid argument form

$$p \rightarrow q,\ p\ \therefore\ q$$

since R is substituted everywhere for $p$, and T is substituted everywhere for $q$. Similarly, the abstract argument

$$\neg \mathrm{E} \rightarrow (\mathrm{B} \vee \mathrm{C}),\ \neg \mathrm{E}\ \therefore\ \mathrm{B} \vee \mathrm{C}$$

is also an instance of this form, with $\neg$E substituted for $p$ and (B ∨ C) for $q$.

An argument is a **substitution instance** of a given argument form if it is obtainable from the form by systematically substituting each occurrence of a given statement variable in the form by the same individual statement, whether simple (e.g., P), or compound (e.g., Q v R).

Thus $\neg \mathrm{E} \rightarrow (\mathrm{B} \vee \mathrm{C}),\ \neg \mathrm{E}\ \therefore\ \mathrm{B} \vee \mathrm{C}$ is a substitution instance of the form $p \rightarrow q,\ p\ \therefore\ q$.

The rule of inference encapsulating the above valid argument form is known by its Latin name, *modus ponens* (the mood that affirms the antecedent).[7]

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[7] Actually, the full name of this rule is *modus ponendo ponens*, "the mood affirming [the second term] by affirming [the first]." As we shall see, this contrasts with *modus tollendo tollens*, "the mood denying [the first term] by denying [the second]," as well as the two forms of the disjunctive syllogism recognized by the Stoics, which the medieval scholars called *modus tollendo ponens*, and *modus ponendo tollens*.

> ***Modus Ponens* (MP)**
> From a conditional statement and its antecedent, infer the consequent.
> *In symbols:*
> From $p \rightarrow q$ and $p$, infer $q$.

This rule of inference is so fundamental and so obvious that it is virtually never explicitly appealed to in natural reasoning, except possibly when you are really beating an illogical opponent over the head with the illogic of his reasoning. But we can't get very far with more complex arguments unless we include it among the basics. Once we have basic rules of inference like this, though, we can prove the validity of more complex arguments. This is the idea behind *natural deduction*, where we set up formal proofs in the style of geometrical proofs. Let's look at an example.

Imagine a Buddhist arguing with a Sãnkhya as follows:

> If qualities are OBSERVABLE then they must be REAL. So you should accept that qualities are THINGS, since you accept that if they are real they are things.

This is an enthymeme with a suppressed premise "qualities are observable." We symbolize it as follows:

$$O \rightarrow R, R \rightarrow T, O \therefore T$$

The idea of a formal proof is simple: we aim to derive the conclusion on the last line. First we state the premises on separate lines, then any subsequent line is derived from those above it by applying our rules of inference:

| | |
|---|---|
| (1) $O \rightarrow R$ | Prem |
| (2) $R \rightarrow T$ | Prem |
| (3) O | Prem |
| (4) R | 1, 3 MP |
| (5) T | 2, 4 MP |

Notice that in the right-hand column we give the justification for each line. The premises are labelled 'Prem.' Line 4 is obtained from lines 1 and 3 by an application of Modus Ponens, and line 5 is similarly obtained from lines 2 and 4. Tacitly, we are applying a procedure first identified by the Stoics, called the *dialectical rule*: "if we have premises that yield a conclusion, then we have in effect also the conclusion among the premises, even if it is not explicitly stated." So far as we know, the Stoics did not set up formal proofs, but instead proved the validity of other argument forms schematically by repeatedly applying this rule together with their five basic rules of inference. Clearly, this amounts to the same thing. A proof such as the one above proves the formal validity of every inference from the premises to a succeeding line derived by valid rules of inference, and therefore the formal validity of an argument from the premises to the conclusion of the last line.

### 3.3.2 AFFIRMING THE CONSEQUENT

In the witch scene from *Monty Python and the Holy Grail*, already encountered in chapter 2, a crucial part of Sir Bedevere's reasoning is this:

> *Bedevere:* Tell me, what do you do with witches?
> *Crowd:* Burn them!
> *Bedevere:* And what do you burn apart from witches?
> *Crowd:* More witches! . . . [pregnant pause]. . . . Wood?
> *Bedevere:* So why do witches burn?
> *Crowd:* 'Cos they're made of wood?
> *Bedevere:* Good!

Sir Bedevere seems to be encouraging the crowd to reason along these lines:

> If witches are made of wood, they'll burn. Witches burn. Therefore they're made of wood.

Obviously, this is invalid reasoning, as we can see by considering another argument of the same form:

> If airliners were lighter than air, they'd fly above the ground. Airliners fly above the ground. Therefore they're lighter than air.

This argument has the same form as Sir Bedevere's. All its premises are true, yet its conclusion is false. So the form is invalid. Hence, any argument that has this form (but is not also an instance of some other valid form) is invalid. This is the case for Sir Bedevere's argument. We can also see it is invalid by applying the root definition of validity: even if we were to accept all its premises, this would still be compatible with denying the conclusion.

This invalid form of argument is beguilingly similar to modus ponens. In the latter we *affirm the antecedent* of the conditional in order to infer its consequent; here we *affirm the consequent* of the conditional in order to make the faulty inference to its antecedent. Hence the name of the fallacy:

| |
|---|
| *Fallacy of Affirming the Consequent (FAC)* |
| $p \rightarrow q, q \therefore p$      *INVALID!* |

The *Python* argument is intended for humorous effect. Yet the mistake is embarrassingly common. Here's a somewhat controversial example. Few scientists have reasoned with the same kind of logical rigour as Sir Isaac Newton. Nevertheless, he appears to have been guilty of something like this fallacy on one occasion. When Edmond Halley (of comet fame) came from London to visit the reclusive professor in his rooms at Cambridge in 1684, he asked him what curve a planet would describe if it was attracted to the Sun by a force reciprocal to the square of its distance. Newton immediately replied that it would be an ellipse, but, on failing to find his calculation among his papers, promised to redo the calculation and send it to Halley. The end result was perhaps the greatest scientific classic of all time: Newton's *Principia*. But as his opponents pointed out to his embarrassment, what Newton proved in the first edition of his *Principia* was that if the curve was an ellipse, the law of force would be the inverse square law—the converse of what Halley had asked for. In effect Newton was arguing

> If the curve is an ellipse, the law of force will be the inverse square law. Thus given Halley's assumption that the law of force is the inverse square law, it follows that the curve is an ellipse.

This fallacy also occurs (all too often) in formal proofs done on autopilot—like this:

| | | |
|---|---|---|
| (1) F → G | Prem | |
| (2) F → H | Prem | |
| (3) H | Prem | |
| (4) F | 2, 3 MP | *ERROR! This is the fallacy FAC!* |
| (5) G | 1, 4 MP | |

SUMMARY

- In stating **argument forms** we use placeholders for any individual statements called **statement variables**, denoted by the lower case letters *p*, *q*, etc. By analogy with the variables in algebra, they stand for **any** statements, whereas the capital letters we have used to represent individual statements are analogous to the particular values of the variables in algebra.
- An argument is a **substitution instance** of a given argument form if it is obtainable from the form by systematically substituting each occurrence of a given statement variable in the form by the same individual statement, whether simple (e.g., P), or compound (e.g., Q ∨ R). Thus ¬ E → (B ∨ C), ¬ E ∴ B ∨ C is a substitution instance of the form $p \rightarrow q, p \therefore q$.
- The rule of inference **modus ponens** (**MP**) is
  From $p \rightarrow q$ and $p$, infer $q$.
  From a conditional statement and its antecedent, infer the consequent.

- The **validity** of this argument form follows from our definition of formal validity: it is impossible for $q$ to be false if $p \rightarrow q$ and $p$ are both true.
- The argument form $p \rightarrow q,\ q\ \therefore\ p$ is INVALID, and is known as the **fallacy of affirming the consequent (FAC)**.

## EXERCISES 3.3

8. *Prove the validity of the* "How do you know she's a witch?" *argument* from Monty Python's *Holy Grail*, using the symbolization suggested*:*

   If she's MADE of wood, she's a WITCH. If she weighs the same as a DUCK, she's made of wood. She weighs the same as a duck. Therefore, she's a witch.

*Prove the validity of the following abstract arguments:*

9. A, A → (B → C), B ∴ C

10. A, A → (A → ¬B) ∴ ¬B

11. (P & Q) → R, P & Q, R → ¬S ∴ ¬S

12. ¬P → (Q ∨ R), (Q ∨ R) → S, ¬P ∴ S

13. ¬(F & G) → (Q ∨ R), ¬P → ¬(F & G), ¬P ∴ Q ∨ R

14. F → G, (F → G) → H ∴ H

15. (F → G) → H, H → F, F → G ∴ G

16. (F → G) → (G → P), P → Q, F → G, F ∴ Q

17. President Clinton, breaking a long silence over the atrocities in East Timor, was quoted as saying:

    "It would be a PITY if the Indonesian recovery were CRUSHED by this. But one way or the other, it will be crushed by this if they don't FIX it."—*The Boston Globe*, Sept 10, 1999

    By symbolizing and constructing a formal proof, show what follows from the President's remarks if one adds the assumption that "they don't fix it."