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SETS

1. Collections of Things

As the 19th century was coming to a close, many people began to try to think precisely about collections. Among the first was the Russian-German mathematician Georg Cantor. Cantor introduced the idea of a *set*. For Cantor, a set is the collection of many things into a whole (1955: 85). It's not hard to find examples of sets: a crowd of people is a set of people, a herd of cows is a set of cows, a fleet of ships is a set of ships, and so on. The things that are collected together into a set are the *members* of the set. So if a library is a set of books, then the books in the library are the members of the library. Likewise, if a galaxy of stars is a set of stars, then the stars in the galaxy are the members of the galaxy.

As time went by, Cantor's early work on sets quickly became an elaborate theory. Set theory went through a turbulent childhood (see van Heijenoort, 1967; Hallett, 1988). But by the middle of the 20th century, set theory had become stable and mature. Set theory today is a sophisticated branch of mathematics. Set theorists have developed a rich and complex technical vocabulary – a network of special terms. And they have developed a rich and complex system of rules for the correct use of those terms. Our purpose in this chapter is to introduce you to the vocabulary and rules of set theory. Why study set theory? Because it is used extensively in current philosophy. You need to know it.

Our approach to sets is uncritical. We take the words of the set theorists at face value. If they say some sets exist, we believe them. Of course, as philosophers, we have to look critically at the ideas behind set theory. We need to ask many questions about the assumptions of the set theorists. But before you can criticize set theory, you need to understand it. We are concerned here only with the understanding. You may or may not think that numbers exist. But you still need to know how to do arithmetic. Likewise, you may or may not think that sets exist. But to succeed in contemporary philosophy, you need to know at least some elementary set theory. Our goal is to help you master the set theory you need to do philosophy. Our approach to sets is informal. We introduce the notions of set theory step by step, little by little. A more formal approach involves the detailed study of the *axioms* of set theory. The axioms of set theory are the precisely stated rules of set theory. Studying the axioms of set theory is advanced work. So we won't go into the axioms here. Our aim is to *introduce* set theory. We can introduce it informally. Most importantly, in the coming chapters, we'll show how ideas from set theory (and other parts of mathematics) are applied in various branches of philosophy.

We start with the things that go into sets. After all, we can't have collections of things if we don't have any things to collect. We start with things that aren't sets. An *individual* is any thing that isn't a set. Sometimes individuals are known as *urelemente* (this is a German word pronounced oor-ella-mentuh, meaning *primordial, basic, original elements*). Beyond saying that individuals are not sets, we place no restrictions on the individuals. The individuals that can go into sets can be names, concepts, physical things, numbers, monads, angels, propositions, possible worlds, or whatever you want to think or talk about. So long as they aren't sets. Sets can be inside sets, but then they aren't counted as *individuals*. Given some individuals, we can collect them together to make sets. Of course, at some point we'll have to abandon the idea that every set is a construction of things collected together by someone. For example, set theorists say that there exists a set whose members are all finite numbers. But no human person has ever gathered all the finite numbers together into a set. Still, for the moment, we'll use that kind of constructive talk freely.

2. Sets and Members

Sets have names. One way to refer to a set is to list its members between curly braces. Hence the name of the set consisting of Socrates, Plato, and Aristotle is {Socrates, Plato, Aristotle}. Notice that listing the members is different from listing the names of the members. So

{Socrates, Plato, Aristotle} is a set of *philosophers*; but

{“Socrates”, “Plato”, “Aristotle”} is a set of *names* of philosophers.

The membership relation is expressed by the symbol $\in$. So we symbolize the fact that Socrates is a member of {Socrates, Plato} like this:

Socrates $\in$ {Socrates, Plato}.

The negation of the membership relation is expressed by the symbol $\notin$. We therefore symbolize the fact that Aristotle is not a member of {Socrates, Plato} like this:

Aristotle $\notin$ {Socrates, Plato}.

And we said that individuals don't have members (in other words, no object is a member of any non-set). So Socrates is not a member of Plato. Write it like this:

Socrates $\notin$ Plato.

Identity. Two sets are identical if, and only if, they have the same members. The long phrase “if and only if” indicates logical equivalence. To say a set S is identical with a set T is equivalent to saying that S and T have the same members. That is, if S and T have the same members, then they're identical; and if S and T are identical, then they have the same members. The phrase "if and only if" is often abbreviated as "iff". It's not a spelling mistake! Thus

S = T if and only if S and T have the same members

is abbreviated as

S = T iff S and T have the same members.

More precisely, a set S is identical with a set T iff for every x, x is in S iff x is in T. One of our goals is to help you get familiar with the symbolism of set theory. So we can write the identity relation between sets in symbols like this:

S = T iff (for every x)(($x \in$ S) iff ($x \in$ T)).

You can easily see that {Socrates, Plato} = {Socrates, Plato}. When writing the name of a set, the order in which the members are listed makes no difference. For example,

{Socrates, Plato} = {Plato, Socrates}.

When writing the name of a set, mentioning a member many times makes no difference. You only need to mention each member once. For example,

{Plato, Plato, Plato} = {Plato, Plato} = {Plato};

{Socrates, Plato, Socrates} = {Socrates, Plato}.

When writing the name of a set, using different names for the same members makes no difference. As we all know, Superman is Clark Kent and Batman is Bruce Wayne. So

{Superman, Clark Kent} = {Clark Kent} = {Superman};

{Superman, Batman} = {Clark Kent, Bruce Wayne}.

Two sets are distinct if, and only if, they have distinct members:

{Socrates, Plato} is not identical with {Socrates, Aristotle}.

3. Set Builder Notation

So far we've defined sets by listing their members. We can also define a set by giving a formula that is true of every member of the set. For instance, consider the set of happy things. Every member in that set is happy. It is the set of all *x* such that *x* is happy. We use a special notation to describe this set:

the set of . . .	{ . . . }
the set of all *x* . . .	{ *x* . . . }
the set of all *x* such that . . .	{ *x* | . . . }
the set of all *x* such that *x* is happy	{ *x* | *x* is happy }.

Note that we use the vertical stroke "|" to mean "such that". And when we use the variable *x* by itself in the set builder notation, the scope of that variable is wide open – *x* can be *anything*. Many sets can be defined using this set-builder notation:

the books in the library = { *x* | *x* is a book and *x* is in the library };

the sons of rich men = { *x* | *x* is the son of some rich man }.

A set is never a member of itself. At least not in standard set theory. There are some non-standard theories that allow sets to be members of themselves (see Aczel, 1988). But we're developing standard set theory here. And since standard set theory is used to define the non-standard set theories, you need to start with it anyway! Any definition of a set that implies that it is a member of itself is ill-formed – it does not in fact define any set at all. For example, consider the formula

the set of all sets = { *x* | *x* is a set }.

Since the set of all sets is a set, it must be a member of itself. But we've ruled out such ill-formed collections. A set that is a member of itself is a kind of vicious circularity. The rules of set theory forbid the formation of any set that is a member of itself. Perhaps there is a *collection* of all sets. But such a collection can't be a *set*.

4. Subsets

Subset. Sets stand to one another in various relations. One of the most basic relations is the subset relation. A set S *is a subset of* a set T iff every member of S is in T. More precisely, a set S is a subset of a set T iff for every *x*, if *x* is in S, then *x* is in T. Hence

{Socrates, Plato} is a subset of {Socrates, Plato, Aristotle}.

Set theorists use a special symbol to indicate that S is a subset of T:

S $\subseteq$ T means S is a subset of T.

Hence

{Socrates, Plato} $\subseteq$ {Socrates, Plato, Aristotle}.

We can use symbols to define the subset relation like this:

S $\subseteq$ T iff (for every x)(if $x \in$ S then $x \in$ T).

Obviously, if x is in S, then x is in S; hence every set is a subset of itself. That is, for any set S, S $\subseteq$ S. For example,

{Socrates, Plato} is a subset of {Socrates, Plato}.

But remember that no set is a member of itself. Being a subset of S is different from being a member of S. The fact that S $\subseteq$ S does *not* imply that S $\in$ S.

Proper Subset. We often want to talk about the subsets of S that are distinct from S. A subset of S that is not S itself is a *proper* subset of S. An identical subset is an *improper* subset. So

{Socrates, Plato} is an improper subset of {Socrates, Plato};

while

{Socrates, Plato} is a proper subset of {Socrates, Plato, Aristotle}.

We use a special symbol to distinguish proper subsets:

S $\subset$ T means S is a proper subset of T.

Every proper subset is a subset. So if S $\subset$ T, then S $\subseteq$ T. However, not every subset is a proper subset. So if S $\subseteq$ T, it does not follow that S $\subset$ T. Consider:

{Socrates, Plato} $\subseteq$ {Socrates, Plato, Aristotle}	True
{Socrates, Plato} $\subset$ {Socrates, Plato, Aristotle}	True
{Socrates, Plato, Aristotle} $\subseteq$ {Socrates, Plato, Aristotle}	True
{Socrates, Plato, Aristotle} $\subset$ {Socrates, Plato, Aristotle}	False

Two sets are identical iff each is a subset of the other:

S = T iff ((S $\subseteq$ T) & (T $\subseteq$ S)).

Superset. A superset is the converse of a subset. If S is a subset of T, then T is a superset of S. We write it like this: T ⊇ S means T is a superset of S. For example,

{Socrates, Plato, Aristotle} ⊇ {Socrates, Plato}.

5. Small Sets

Unit Sets. Some sets contain one and only one member. A *unit set* is a set that contains exactly one member. For instance, the unit set of Socrates contains Socrates and only Socrates. The unit set of Socrates is {Socrates}. For any thing x, the unit set of x is $\{x\}$. Sometimes unit sets are known as *singleton* sets. Thus {Socrates} is a singleton set.

Some philosophers have worried about the existence of unit sets (see Goodman, 1956; Lewis, 1991: sec 2.1). Nevertheless, from our uncritical point of view, these worries aren't our concern. Set theorists tell us that there are unit sets, so we accept their word uncritically. They tell us that for every x, there exists a unit set $\{x\}$. And they tell us that x is not identical to $\{x\}$. On the contrary, $\{x\}$ is distinct from x. For example, {Socrates} is distinct from Socrates. Socrates is a person; {Socrates} is a set. Consider the set of all x such that x is a philosopher who wrote the *Monadology*. This is written formally as:

{ x | x is a philosopher who wrote the *Monadology* }.

Assuming that Leibniz is the one and only philosopher who wrote the *Monadology*, then it follows that this set is {Leibniz}.

Empty Set. The set of all x such that x is a dog is { x | x is a dog }. No doubt this is a large set with many interesting members. But what about the set of all x such that x is a unicorn? Since there are no unicorns, this set does not have any members. Or at least the set of actual unicorns has no members. Likewise, the set of all actual elves has no members. So the set of all actual unicorns is identical to the set of all actual elves.

A set that does not contain any members is said to be *empty*. Since sets are identified by their members, there is exactly one empty set. More precisely,

S is the empty set iff (for every x)(x is not a member of S).

Two symbols are commonly used to refer to the empty set:

∅ is the empty set; and
{} is the empty set.

We'll use "{}" to denote the empty set. For example,

{} = { *x* | *x* is an actual unicorn };
{} = { *x* | *x* is a married bachelor }.

It is important not to be confused about the empty set. The empty set isn't nothing or non-being. If you think the empty set exists, then obviously you can't think that it is nothing. That would be absurd. The empty set is exactly what the formalism says it is: it is a set that does not contain any thing as a member. It is a set with no members. According to set theory, the empty set is an existing particular object.

The empty set is a subset of every set. Consider an example: {} is a subset of {Plato, Socrates}. The idea is this: for any *x*, if *x* is in {}, then *x* is in {Plato, Socrates}. How can this be? Well, pick some object for *x*. Let *x* be Aristotle. Is Aristotle in {}? The answer is no. So the statement "Aristotle is in {}" is false. And obviously, "Aristotle is in {Plato, Socrates} is false. Aristotle is not in that set. But logic tells us that the only way an *if-then* statement can be false is when the *if* part is true and the *then* part is false. Thus (somewhat at odds with ordinary talk) logicians count an *if-then* statement with a false *if* part as true. So even though both the *if* part and the *then* part of the whole *if-then* statement are false, the whole *if-then* statement "if Aristotle is in {}, then Aristotle is in {Socrates, Plato}" is true. The same reasoning holds for any object you choose for *x*. Thus for any set S, and for any object *x*, the statement "if *x* is in {}, then *x* is in S" is true. Hence {} is a subset of S.

We can work this out more formally. For any set S, {} $\subseteq$ S. Here's the proof: for any *x*, it is not the case that ($x \in$ {}). Recall that when the antecedent (the *if* part) of a conditional is false, the whole conditional is true. That is, for any Q, when P is false, (if P then Q) is true. So for any set S, and for any object *x*, it is true that (if $x \in$ {} then *x* is in S). So for any set S, it is true that (for all *x*)(if $x \in$ {} then *x* is in S). Hence for any set S, {} $\subseteq$ S.

Bear this clearly in mind: the fact that {} is a *subset* of every set does *not* imply that {} is a *member* of every set. The subset relation is *not* the membership relation. Every set has the empty set as a subset. But if we want the empty set to be a member of a set, we have to put it into the set. Thus {A} has the empty set as a subset while {{}, A} has the empty set as both a subset and as a member. Clearly, {A} is not identical to {{}, A}.

6. Unions of Sets

Unions. Given any two sets S and T, we can take their *union*. Informally, you get the union of two sets by adding them together. For instance, if the Greeks = {Socrates, Plato} and the Germans = {Kant, Hegel}, then the union of the

Greeks and the Germans is {Socrates, Plato, Kant, Hegel}. We use a special symbol to indicate unions:

the union of S and T = S union T = S ∪ T.

For example,

{Socrates, Plato} ∪ {Kant, Hegel} = {Socrates, Plato, Kant, Hegel}.

When forming the union of two sets, any common members are only included once:

{Socrates, Plato} ∪ {Plato, Aristotle} = {Socrates, Plato, Aristotle}.

The union of a set with itself is just that very same set:

{Socrates} ∪ {Socrates} = {Socrates}.

More formally, the union of S and T is the set that contains every object that is either in S or in T. Thus *x* is in the union of S and T iff *x* is in S only, or *x* is in T only, or *x* is in both. The union of S and T is the set of all *x* such that *x* is in S or *x* is in T. In symbols,

S union T = { *x* | *x* is in S or *x* is in T };

$S \cup T = \{ x \mid x \in S \text{ or } x \in T \}$.

Just as you can add many numbers together, so you can union many sets together. Just as you can calculate 2+3+6, so you can calculate S ∪ T ∪ Z. For example, {Socrates, Plato} ∪ {Kant, Hegel} ∪ {Plotinus} is {Socrates, Plato, Kant, Hegel, Plotinus}.

When a set is unioned with the empty set, the result is itself: S ∪ {} = S. The union operator is defined only for *sets*. So the union of an individual with a set is undefined as is the union of two individuals.

7. Intersections of Sets

Intersections. Given any two sets S and T, we can take their *intersection*. Informally, you get the intersection of two sets by taking what they have in common. For instance, if the Philosophers = {Socrates, Aristotle} and the Macedonians = {Aristotle, Alexander}, then the intersection of the Philosophers with the Macedonians = {Aristotle}. We use a special symbol to indicate intersections:

the intersection of S and T = S intersection T = S ∩ T.